Math 20550 - Calculus III - Summer 2014
June 20, 2014
Exam 1

Name: Solutions

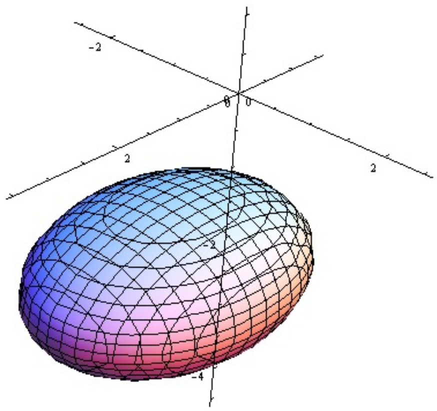
There is no need to use calculators on this exam. This exam consists of 6 problems on 7 pages. You have 50 minutes to work on the exam. There are a total of 65 available points and a perfect score on the exam is 60 points. There will be a lecture after the exam. All electronic devices should be turned off and put away. The only things you are allowed to have are: a writing utensil(s) (pencil preferred), an eraser, and an exam. No notes, books, or any other kind of aid are allowed. All answers should be given as exact, closed form numbers as opposed to decimal approximations (i.e., π as opposed to 3.14159265358979...). You must show all of your work to receive credit. Please box your final answers. Cheating is strictly forbidden. Good luck!

Honor Pledge: As a member of the Notre Dame community, I will not participate in, nor tolorate academic dishonesty. My signature here binds me to the Notre Dame Honor Code:

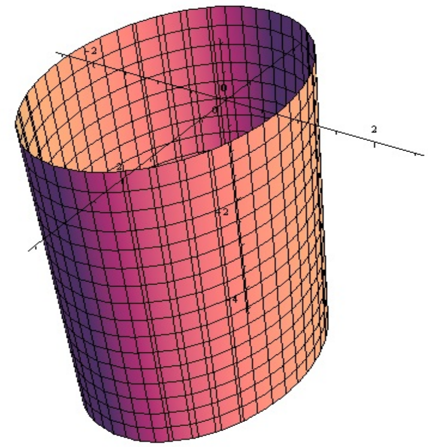
Signature:____________________________________

Problem	Score
1	/10
2	/15
3	/10
4	/10
5	/10
6	/10
Score	/60

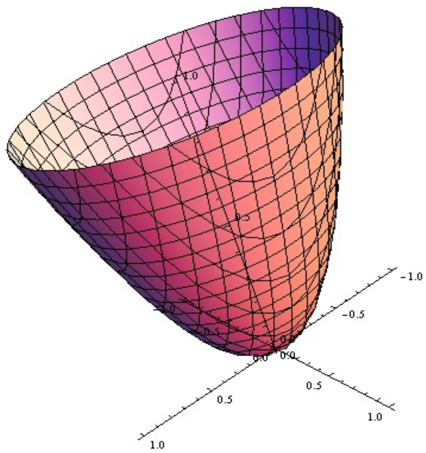
Problem 1 (10 points - 2.5 points each). *Match the equation to the surface*



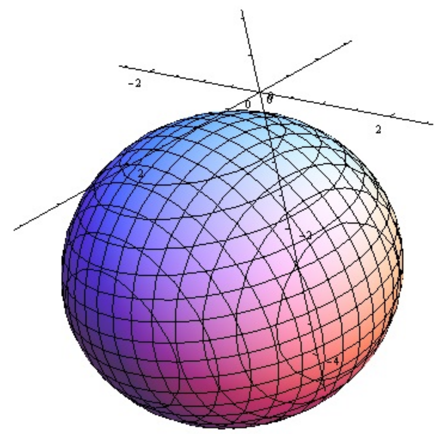
I.



II.



III.



IV.

Write I, II, III, or IV next to the appropriate equation:

(a) ____*III*____ $x^2 + 2y^2 - z = 0$

(b) ____*IV*____ $(x-1)^2 + y^2 + (z+2)^2 = 4$

(c) ____*I*____ $(x-1)^2 + 2y^2 + 4(z+2)^2 = 4$

(d) ____*II*____ $(x-1)^2 + 2y^2 = 4$

Problem 2 (15 points - 5 points each)**.** *Consider the two points $P = (2, 0, 1)$ and $Q = (0, 1, 1)$.*

(a) *Find an equation for the plane which contains Q and is perpendicular to $\overrightarrow{QP}$.*
(b) *Find the distance from this plane to the origin.*
(c) *Give an equation for a plane which passes through Q and is perpendicular to the plane you found in part (a). (There are several correct answers. You only need to give one example.)*

Solution.

(a) The vector $\overrightarrow{QP}$ is given by

$$\overrightarrow{QP} = P - Q = \langle 2-0, 0-1, 1-1\rangle = \langle 2, -1, 0\rangle .$$

This is the normal vector to the plane. Recall that the equation for the plane passing through a point Q with normal vector $\vec{n}$ is

$$\vec{n} \cdot ((x, y, z) - Q) = 0.$$

In this case, $\vec{n} = \overrightarrow{QP}$, so the equation is

$$\langle 2, -1, 0\rangle \cdot \langle x-0, y-1, z-1\rangle = 2(x-0) - (y-1) + 0(z-1) = 2x - y + 1 = 0.$$

The answer can also be written as $2x - y = -1$.

(b) Recall that the distance between a point R and a plane with normal vector $\vec{n}$ containing some point Q is given by

$$d = \frac{\left|\overrightarrow{QR} \cdot \vec{n}\right|}{\|\vec{n}\|}$$

(you could also use $\overrightarrow{RQ}$ since there is an absolute value in the numerator). Here, $R = (0, 0, 0)$, so $\overrightarrow{QR} = \langle 0, -1, -1\rangle$ and $\vec{n} = \langle 2, -1, 0\rangle$. So, the distance from the origin to the plane is

$$d = \frac{|\langle 0, -1, -1\rangle \cdot \langle 2, -1, 0\rangle|}{\|\langle 2, -1, 0\rangle\|} = \frac{|0+1+0|}{\sqrt{4+1+0}} = \frac{1}{\sqrt{5}}$$

(c) We already have a point in the plane, Q, so all we need is a normal vector to the plane. Since the plane must be perpendicular to the plane in part (a), any (nonzero) vector which is perpendicular to $\overrightarrow{QP}$ will suffice (this is why there are several correct answers). There are many ways to find a vector perpendicular to $\overrightarrow{QP}$, two of which are

(1) by observation (looking at the vector $\overrightarrow{QP}$ and choosing another vector accordingly), e.g., looking at $\overrightarrow{QP}$, I can tell that the vector $\mathbf{v} = \langle 1, 2, 0\rangle$ is perpendicular to it. If you do this method, be sure to show it actually is perpendicular! To do this, take the dot product:

$$\overrightarrow{QP} \cdot \mathbf{v} = \langle 2, -1, 0\rangle \cdot \langle 1, 2, 0\rangle = 2 - 2 + 0 = 0.$$

Then, an equation for this perpendicular plane is

$$\langle 1, 2, 0\rangle \cdot \langle x-0, y-1, z-1\rangle = (x-0) + 2(y-1) + 0(z-1) = x + 2y - 2 = 0$$

(2) use the plane from part (a) to find one. The plane in part (a) contains all of the vectors which are perpendicular to $\overrightarrow{QP}$. Take any point R (not equal to Q) in the plane, then the vector $\overrightarrow{QR}$ will be perpendicular to $\overrightarrow{QP}$. For example, the point $R = (0, 1, 0)$ is in the plane (you should verify this by plugging it in to the equation for the plane: $2(0) - (-1) = 1$) so a normal vector is

$$\overrightarrow{QR} = R - Q = \langle 0, 0, -1\rangle$$

and so an equation for a normal plane is

$$\langle 0, 0, -1\rangle \cdot \langle x-0, y-1, z-1\rangle = 0(x-0) + 0(y-1) - (z-1) = -z + 1 = 0.$$

I want to again emphasize that there are several answers to part (c), and so your answer could be different from mine here.

□

Problem 3 (10 points - 5 points each).

(a) *A sphere with center* $(2,1,3)$ *contains a point* $(6,1,3)$. *Find an equation of the sphere.*
(b) *What is the intersection of this sphere with the yz-plane?*

Solution.

(a) First, find the radius of the sphere, which is the distance from the center to any point on the sphere. In this case, the distance from $(2,1,3)$ to $(6,1,3)$ is 4. Thus, an equation for the sphere is

$$(x-2)^2 + (y-1)^2 + (z-3)^2 = 16.$$

(b) To find the intersection with the yz-plane, we simply set $x = 0$ in the equation for the sphere:

$$(0-2)^2 + (y-1)^2 + (z-3)^2 = 4 + (y-1)^2 + (z-3)^2 = 16$$

so the intersection with the yz-plane is

$$(y-1)^2 + (z-3)^2 = 12$$

□

Problem 4 (10 points - 5 points each)**.**

(a) *Find the area of the triangle with vertices* $(0,0,0)$*,* $(1,2,3)$*, and* $(-1,1,-4)$*.*

(b) *Find the volume of the parallelepiped spanned by the vectors*

$$\mathbf{u} = \langle 3,3,2\rangle, \mathbf{v} = \langle 2,0,1\rangle, \textit{ and } \mathbf{w} = \langle 0,-3,0\rangle.$$

Solution.

(a) Let $\mathbf{a}$ be the vector from $(0,0,0)$ to $(1,2,3)$, and let $\mathbf{b}$ be the vector from $(0,0,0)$ to $(-1,1,-4)$. The area of the triangle is half the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. Thus, the area of the triangle is

$$A = \frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|$$

First,

$$\mathbf{a} = \langle 1,2,3\rangle \quad \text{and} \quad \mathbf{b} = \langle -1,1,-4\rangle$$

so

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 1 & -4 \end{vmatrix} = \hat{\imath}(-8-3) - \hat{\jmath}(-4+3) + \hat{k}(1+2) = \langle -11,1,3\rangle$$

Thus, the area of the triangle is

$$A = \frac{1}{2}\|\vec{a}\times\vec{b}\| = \frac{1}{2}\|\langle -11,1,3\rangle\| = \frac{1}{2}\sqrt{121+1+9} = \frac{\sqrt{131}}{2}$$

(b) The volume of the parallelepiped spanned by the vectors

$$\mathbf{u} = \langle 3,3,2\rangle, \mathbf{v} = \langle 2,0,1\rangle, \text{ and } \mathbf{w} = \langle 0,-3,0\rangle,$$

is

$$Vol = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$$

Recall that this triple mixed product is given by a determinant with these three vectors as rows:

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 3 & 3 & 2 \\ 2 & 0 & 1 \\ 0 & -3 & 0 \end{vmatrix} = 3(0+3) - 3(0-0) + 2(-6-0) = 9 - 12 = -3$$

so the volume is

$$Vol = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = |-3| = 3$$

□

Problem 5 (10 points)**.** *Benjamin's Audi broke down and he needs it towed 3km down the street to the mechanic's shop. The tow truck picks up the useless Audi and the tow truck's chain makes an angle of* $60°$ *with the ground. The truck tows the car using a force of 1600N. How much work does the tow truck do in moving the useless Audi down the street?*

Solution. Recall that the energy needed to tow the useless Audi is given by

$$W = \mathbf{F} \cdot \mathbf{d}$$

where $\mathbf{F}$ is the force the tow truck applies to the useless Audi, and $\mathbf{d}$ is the displacement vector of the car. In the statement of the problem, we are only given the magnitudes of the two vectors, but we do know the angle between them as well, so we can use the alternative form of the dot product to find the work done:

$$W = \|\mathbf{F}\|\|\mathbf{d}\| \cos\theta$$

where θ is the angle between $\mathbf{F}$ and $\mathbf{d}$. So, we have that the amount of work the tow truck does is

$$W = (1600N)(3km)\cos(60°) = (4800kJ)(0.5) = 2400kJ.$$

What an awful lot of energy to waste on Benjamin's useless Audi! ☐

Problem 6 (10points). *Determine if the lines*

$$L_1 \quad : \quad x = 4t + 2, \quad y = 3, \qquad\quad z = -t + 1$$

$$L_2 \quad : \quad x = 2s + 2, \quad y = 2s + 3, \quad z = s + 1$$

intersect. If they do, find the point of intersection and the cosine of the angle of their intersection.

Solution. First, to find if they intersect, we set the components of each line equal to each other, and try to solve the system of equations:

$$\begin{cases} 4t + 2 &= 2s + 2 \quad \textcircled{1} \\ 3 &= 2s + 3 \quad \textcircled{2} \\ -t + 1 &= s + 1 \quad \textcircled{3} \end{cases}$$

Equation ② tells us that $s = 0$. Plugging that into equation ①, we see that $t = 0$ as well. Plugging $t = 0$ and $s = 0$ into equation ③ gives $1 = 1$, so this is indeed a solution to the system. This means that the two lines do intersect. To find the point of intersection, plug $t = 0$ into L_1 or $s = 0$ into L_2. We will do the former:

$$(x, y, z) = (4(0) + 2, 3, -(0) + 1) = (2, 3, 1)$$

So, the point of intersection is $(2, 3, 1)$. To find the angle of intersection of the lines is to find the angle between the direction vectors of the lines. We read these off of the coefficients of the variables: for L_1, the direction vector is $\mathbf{v}_1 = \langle 4, 0, -1 \rangle$, and for L_2, the direction vector is $\mathbf{v}_2 = \langle 2, 2, 1 \rangle$. The cosine of the angle between these two vectors is

$$\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|} = \frac{8 + 0 - 1}{\sqrt{16 + 0 + 1}\sqrt{4 + 4 + 1}} = \frac{7}{\sqrt{17}\sqrt{9}} = \frac{7}{3\sqrt{17}}$$

□